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An Upper Bound On the Total Vertex Irregularity Strength of the Cartesian Product of P_2 and an Arbitrary Regular Graph

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Abstract

Let G be a connected and simple graph with vertex set $V(G)$ and edge set $E(G)$. A total labeling $f : V \cup E \to \{1, 2, \ldots, k\}$ is called a *vertex irregular total k-labeling* of G if every two distinct vertices x and y in $V(G)$ satisfy $w_f(x) \neq w_f(y)$, where $w_f(x) = f(x) + \sum_{xz \in E(G)} f(xz)$. The *total vertex irregularity strength* of G, denoted by $tvs(G)$, is the minimum k for which G has a vertex irregular total k-labeling. In this paper, we provide an upper bound on the total vertex irregularity strength of the Cartesian product of P_2 and an arbitrary regular graph G.

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1. Introduction

A vertex irregular total k-labeling was introduced by Bača *et al.* [2]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A total labeling $f : V \cup E \to \{1, 2, \ldots, k\}$ is called a *vertex irregular total k-labeling* of G if every two distinct vertices x and y in $V(G)$ satisfy $w_f(x) \neq w_f(y)$, where $w_f(x) = f(x) + \sum_{xz \in E(G)} f(xz)$. The *total vertex irregularity strength* of G, denoted by $\text{tvs}(G)$, is the minimum k for which G has a vertex irregular total k-labeling.

By $(p, q)-$ graph G we denote a graph G with p order and q size. Bača *et al.* [2] gave a lower and upper bounds on $tvs(G)$ for an arbitrary $(p, q)-$ graph G with the minimum degree δ and the maximum degree Δ as follows:

$$\lceil (p + \delta)/(\Delta + 1) \rceil \leq \text{tvs}(G) \leq p + \Delta - 2\delta + 1. \quad (1)$$

In the same paper, Bača *et al.* [2] proved that

$$\text{tvs}(G) \leq p - 1 - \lfloor (p - 2)/(\Delta + 1) \rfloor \text{ for a } (p, q) - \text{graph } G \text{ with no component of order } \leq 2. \quad (2)$$

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An upper bound on $tvs(G)$ was given also by Przybylo[7]. In the paper, it was proved that

$$\text{tvs}(G) < 32p/\delta + 8 \text{ for an arbitrary graph } G \tag{3}$$

and

$$\text{tvs}(G) < 8p/r + 3 \text{ for an arbitrary } r-\text{regular graph.} \tag{4}$$

This result was improved by Anholcer *et al.*[1]. The new result is

$$\text{tvs}(G) \le 3\lceil p/\delta \rceil + 1 \le 3p/\delta + 4. \tag{5}$$

Ramdani *et al.*[8] gave an upper bound on the total vertex irregularity strength for a disjoint union of arbitrary graphs $\bigcup_{i=1}^{m} G_i$ as follows :

Let G_i be an r-regular graph of order p_i and size q_i, for $i = 1, 2, \ldots, m$. Then,

$$tvs\left(\bigcup_{i=1}^{m} G_i\right) \le \sum_{i=1}^{m} tvs(G_i) - \left\lfloor \frac{m-1}{2} \right\rfloor. \tag{6}$$

In the same paper, Ramdani *et al.*[8] also gave an exact value of $tvs\left(\bigcup_{i=1}^{m} G_i\right)$, which is

$$tvs\left(\bigcup_{i=1}^{m} G_i\right) = \sum_{i=1}^{m} tvs(G_i) - m + 1 \tag{7}$$

if there is a vertex irregular total $(tvs(G_i))$-labeling $f_i : V(G_i) \cup E(G_i) \to \{1, 2, \ldots, tvs(G_i)\}$ of G_i such that the vertex-weight function $w_{f_i}(v_{ia}) : V(G_i) \to \{r+1, r+2, \ldots, (r+1)tvs(G_i) - 1\}$ is a bijection for every $i = 1, 2, \ldots, m$.
Nurdin *et al.*[4] gave a lower bound on $tvs(G)$ as follows :

$$tvs(G) \ge \max\left\{ \left\lceil \frac{\delta + n_\delta}{\delta + 1} \right\rceil, \left\lceil \frac{\delta + n_\delta + n_{\delta+1}}{\delta + 2} \right\rceil, \ldots, \left\lceil \frac{\delta + \sum_{i=\delta}^{\Delta} n_i}{\Delta + 1} \right\rceil \right\}, \tag{8}$$

where n_i denotes the number of vertices of degree i for every $i \in \{\delta, \delta + 1, \ldots, \Delta\}$.

Bača *et al.*[2] gave an exact value of the total vertex irregularity strength of a complete graph with order p as follows. Let K_p be a complete graph with order p, then

$$tvs(K_p) = 2 \text{ for } p \ge 2. \tag{9}$$

Some other results about the total vertex irregularity strength of a graph G were given by Majerski *et al.*[3], Nurdin *et al.*[5], [6], and Wijaya *et al.*[9], [10].

2. Main Results

Definition 1. *The Cartesian product $G\Box H$ of graphs G and H is a graph such that the vertex set of $G\Box H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices (u, u') and (v, v') are adjacent in $G\Box H$ if and only if either $u = v$ and u' is adjacent with v' in H, or $u' = v'$ and u is adjacent with v in G.*

The first result below gives an upper bound on the total vertex irregularity strength of the Cartesian product graph of P_2 and G for an arbitrary regular graph G.

Theorem 2. *Let G be an $r-$regular graph for $r \ge 1$. Then*

$$tvs(P_2 \Box G) \le 2tvs(G).$$

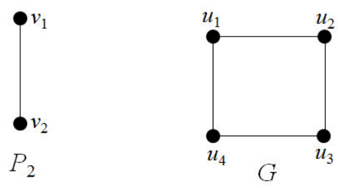

Fig. 1. A path P_2 and a graph G

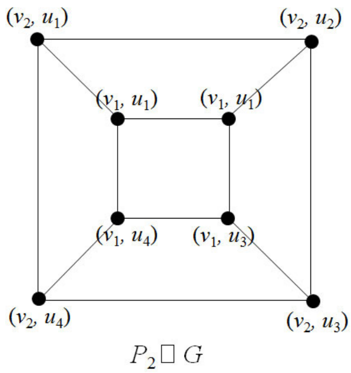

Fig. 2. The Cartesian product of P_2 and G

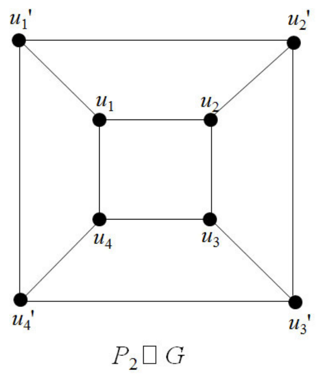

Fig. 3. The notation of the vertices of $P_2\Box G$ defined by Theorem 2

Proof. Let G be an $r-$regular graph with order n, size e, and $tvs(G) = t$. Let $V(G) = \{u_1, u_2, \cdots, u_n\}$ and $V(P_2) = \{v_1, v_2\}$.

Denote $V(P_2\Box G) = \{u_1, u_2, \cdots u_n\} \cup \{u'_1, u'_2, \cdots, u'_n\}$, where $u_i = (v_1, u_i)$ and $u'_i = (v_2, u_i)$ for $1 \leq i \leq n$.

An illustration of the notation of the vertices of $P_2\Box G$ defined by Theorem 2 is given in the figure below.

Let f be a vertex irregular total t-labeling of G. Define a total $2t$-labeling g of $P_2\Box G$ as follows:

$$\begin{aligned}
g(u_i) &= f(u_i);\\
g(u'_i) &= f(u_i) + t - 1;\\
g(u_iu_j) &= f(u_iu_j);\\
g(u'_iu'_j) &= f(u_iu_j) + t;\\
g(u_iu'_i) &= 1.
\end{aligned}$$

It will be shown that there are no two vertices in $P_2 \Box G$ of the same weight.

Let u_x and u_y be different vertices in G. Let the vertices adjacent to u_x in G be $u_{x1}, u_{x2}, \cdots, u_{xr}$ and the vertices adjacent to u_y in G be $u_{y1}, u_{y2}, \cdots, u_{yr}$. We consider 3 cases below.

1 It will be shown that $w_g(u_x) \neq w_g(u_y)$.

$$\begin{aligned} w_g(u_x) &= g(u_x) + \textstyle\sum_{s=1}^{r} g(u_x u_{xs}) + g(u_x u'_x) \\ &= f(u_x) + \textstyle\sum_{s=1}^{r} f(u_x u_{xs}) + 1 \\ &= w_f(u_x) + 1; \end{aligned}$$

$$\begin{aligned} w_g(u_y) &= g(u_y) + \textstyle\sum_{s=1}^{r} g(u_y u_{ys}) + g(u_y u'_y) \\ &= f(u_y) + \textstyle\sum_{s=1}^{r} f(u_y u_{ys}) + 1 \\ &= w_f(u_y) + 1. \end{aligned}$$

Since $w_f(u_x) \neq w_f(u_y)$ for every $x \neq y$, we have $w_g(u_x) \neq w_g(u_y)$ for every $x \neq y$.

2 It will be shown that $w_g(u'_x) \neq w_g(u'_y)$.

$$\begin{aligned} w_g(u'_x) &= g(u'_x) + \textstyle\sum_{s=1}^{r} g(u'_x u'_{xs}) + g(u_x u_{x'}) \\ &= f(u_x) + t - 1 + \textstyle\sum_{s=1}^{r} (f(u_x u_{xs}) + t) + 1 \\ &= f(u_x) + t - 1 + \textstyle\sum_{s=1}^{r} f(u_x u_{xs}) + tr + 1 \\ &= f(u_x) + \textstyle\sum_{s=1}^{r} f(u_x u_{xs}) + t(r+1) \\ &= w_f(u_x) + t(r+1); \end{aligned}$$

$$\begin{aligned} w_g(u'_y) &= g(u'_y) + \textstyle\sum_{s=1}^{r} g(u'_y u'_{ys}) + g(u_y u_{y'}) \\ &= f(u_y) + t - 1 + \textstyle\sum_{s=1}^{r} \Big(f(u_y u_{ys}) + t\Big) + 1 \\ &= f(u_y) + t - 1 + \textstyle\sum_{s=1}^{r} f(u_y u_{ys}) + tr + 1 \\ &= f(u_y) + \textstyle\sum_{s=1}^{r} f(u_y u_{ys}) + t(r+1) \\ &= w_f(u_y) + t(r+1). \end{aligned}$$

Since $w_f(u_x) \neq w_f(u_y)$ for every $x \neq y$ and $t(r+1)$ is a constant, we get $w_g(u_x) \neq w_g(u_y)$ for every $x \neq y$.

3 It will be shown that $w_g(u_x) \neq w_g(u'_y)$.
From the first case above, we get

$$w_g(u_x) = f(u_x) + \textstyle\sum_{s=1}^{r} f(u_x u_{xs}) + 1.$$

Because the maximum label in the labeling f is t and every vertex in G has degree r, we get

$$w_g(u_x) \leq t(r+1) + 1.$$

Furthermore, from the second case above, we have

$$w_g(u'_y) = f(u_y) + \textstyle\sum_{s=1}^{r} f(u_y u_{ys}) + t(r+1).$$

Since the the minimum label in the labeling f is 1 and $r \geq 1$, we have $f(u_y) + \sum_{s=1}^{r} f(u_y u_{ys}) \geq 2$. So, we get

$$w_g(u'_y) \geq 2 + t(r+1) > t(r+1) + 1.$$

Therefore, $w_g(u_x) \neq w_g(u'_y)$.

From the three cases above, there are no two vertices in $P_2 \Box G$ of the same weight. Therefore, g is a vertex irregular total $2tvs(G)$-labeling of $P_2 \Box G$. We conclude that $tvs(P_2 \Box G) \leq 2tvs(G)$. □

The second theorem gives an upper bound on the total vertex irregularity strength of the Cartesian product of P_2 and certain graph G.

Theorem 3. *Let G be an r-regular graph of order p and size q. If there is a vertex irregular total $(tvs(G))$-labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, tvs(G)\}$ of G such that the $w_f(u) < (r+1)tvs(G)$ for every $u \in V(G)$, then*

$$tvs\,(P_2 \Box G) \leq 2tvs(G) - 1.$$

Proof. Let G be an r-regular graph of order p, size q, and $tvs(G) = t$. Let $V(G) = \{u_1, u_2, \cdots, u_n\}$ and $V(P_2) = \{v_1, v_2\}$. Let $V(P_2 \Box G) = \{u_1, u_2, \cdots u_n\} \cup \{u'_1, u'_2, \cdots, u'_n\}$ where $u_i = (v_1, u_i)$ and $u'_i = (v_2, u_i)$ for $1 \leq i \leq n$.

The illustration of the notation of the vertices of $P_2 \Box G$ can be seen in Figures 2 and 3.

Let f be a vertex irregular total t-labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, t\}$ of G such that the $w_f(u) < (r+1)t$ for every $u \in V(G)$.

Define a total $(2t-1)$-labeling g of $P_2 \Box G$ as follows:

$$\begin{aligned}
g(u_i) &= f(u_i);\\
g(u'_i) &= f(u_i) + t - 1;\\
g(u_iu_j) &= f(u_iu_j);\\
g(u'_iu'_j) &= f(u_iu_j) + t - 1;\\
g(u_iu'_i) &= 1.
\end{aligned}$$

It will be shown that there are no two vertices in $P_2 \Box G$ of the same weight.

Let u_x and u_y be different vertices in G. Let the vertices adjacent to u_x in G are $u_{x1}, u_{x2}, \cdots, u_{xr}$ and the vertices adjacent to u_y in G are $u_{y1}, u_{y2}, \cdots, u_{yr}$. As in the proof of Theorem 2, we will consider 3 cases, when the first two cases will be done similarly.

1 It will be shown that $w_g(u_x) \neq w_g(u_y)$.

$$\begin{aligned}
w_g(u_x) &= g(u_x) + \textstyle\sum_{s=1}^{r} g(u_xu_{xs}) + g(u_xu_{x'})\\
&= f(u_x) + \textstyle\sum_{s=1}^{r} f(u_xu_{xs}) + 1\\
&= w_f(u_x) + 1;
\end{aligned}$$

$$\begin{aligned}
w_g(u_y) &= g(u_y) + \textstyle\sum_{s=1}^{r} g(u_yu_{ys}) + g(u_yu_{y'})\\
&= f(u_y) + \textstyle\sum_{s=1}^{r} f(u_yu_{ys}) + 1\\
&= w_f(u_y) + 1.
\end{aligned}$$

Since $w_f(u_x) \neq w_f(u_y)$ for every $x \neq y$, we get $w_g(u_x) \neq w_g(u_y)$ for every $x \neq y$.

2 It will be shown that $w_g(u'_x) \neq w_g(u'_y)$.

$$\begin{aligned}
w_g(u'_x) &= g(u'_x) + \textstyle\sum_{s=1}^{r} g(u'_xu'_{xs}) + g(u_xu_{x'})\\
&= f(u_x) + t - 1 + \textstyle\sum_{s=1}^{r} (f(u_xu_{xs}) + t - 1) + 1\\
&= f(u_x) + t - 1 + \textstyle\sum_{s=1}^{r} f(u_xu_{xs}) + tr - r + 1\\
&= f(u_x) + \textstyle\sum_{s=1}^{r} f(u_xu_{xs}) + t(r+1) - r\\
&= w_f(u_x) + t(r+1) - r;
\end{aligned}$$

$$\begin{aligned}
w_g(u'_y) &= g(u'_y) + \textstyle\sum_{s=1}^{r} g(u'_yu'_{ys}) + g(u_yu_{y'})\\
&= f(u_y) + t - 1 + \textstyle\sum_{s=1}^{r} \Big(f(u_yu_{ys}) + t - 1\Big) + 1\\
&= f(u_y) + t - 1 + \textstyle\sum_{s=1}^{r} f(u_yu_{ys}) + tr - r + 1\\
&= w_f(u_y) + t(r+1) - r.
\end{aligned}$$

Since $w_f(u_x) \neq w_f(u_y)$ for every $x \neq y$ and $t(r+1) - r$ is a constant, we have $w_g(u_x) \neq w_g(u_y)$ for every $x \neq y$.

3 It will be shown that $w_g(u_x) \neq w_g(u'_y)$.
From case 1 above, we get

$$w_g(u_x) = w_f(u_x) + 1$$

and from case 2 above, we get

$$w_g(u'_y) = w_f(u_y) + t(r+1) - r.$$

Since the weight of each vertex in $P_2\Box G$ in the labeling f at least $r+1$ and the weight of each vertex in $P_2\Box G$ in the labeling f is less than $t(r+1)$, we get

$$w_f(u_y) \geq r+1$$

and

$$w_f(u_x) \leq t(r+1) - 1.$$

Therefore,

$$w_g(u_x) < t(r+1) + 1 \leq r + 1 + t(r+1) - r \leq w_f(u_y) + t(r+1) - r \leq w_g(u'_y).$$

So that, $w_g(u_x) \neq w_g(u'_y)$.

From the three cases above, there are no two vertices in $P_2\Box G$ of the same weight. Therefore, g is a vertex irregular total $(2t-1)-$labeling of $P_2\Box G$. We conclude that

$$tvs(P_2\Box G) \leq 2t - 1.$$

□

Now, we will show that the upper bound given in Theorem 3 is tight, by proving $tvs(P_2\Box K_n) = 2tvs(K_n) - 1$, for $n \geq 3$.

Bača *et al.* [2] gave a total vertex irregular-2 labeling φ of K_n for $n \geq 2$ with $V(K_n) = \{v_1, v_2, \cdots, v_n\}$ as follows:

$$\varphi(v_i) = 1 \text{ for } 1 \leq i \leq \left\lceil \tfrac{n}{2} \right\rceil;$$
$$\varphi(v_i) = 2 \text{ for } \left\lceil \tfrac{n}{2} \right\rceil < i \leq n;$$

and for every i, $1 \leq i \leq n$,

$$\varphi(v_iv_j) = 1 \text{ for } 1 \leq j \leq n - i + 1,\ i \neq j;$$
$$\varphi(v_iv_j) = 2 \text{ for } n - i + 2 \leq j \leq n.$$

An illustration of the labeling φ can be seen in Fig. 4.

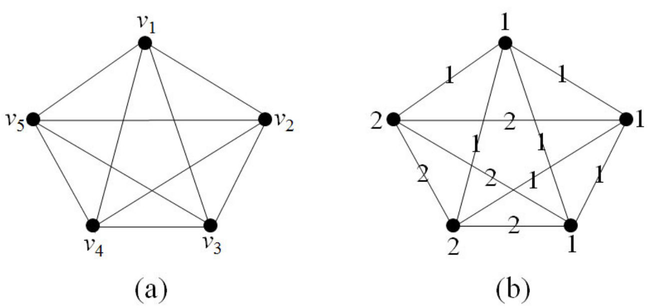

Fig. 4. (a) A complete graph K_5; (b) A total vertex irregular 2−labeling φ of K_5

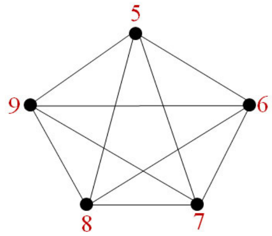

Fig. 5. The weight of vertices of K_5

The labeling φ gives the weight of vertices in K_5 as follows :

$$w_\varphi(v_1) = 5;\ w_\varphi(v_2) = 6;\ w_\varphi(v_3) = 7;\ w_\varphi(v_4) = 8;\ w_\varphi(v_5) = 9.$$

The illustration of the weight of vertices in K_5 is given below.

It is clear that $w_\varphi(v_i) < 2n$ for every $v_i \in V(K_n)$. So that, from Theorem 3, we get an upper bound on $tvs(P_2 \Box K_n)$ is $2tvs(K_n) - 1 = 2 \cdot 2 - 1 = 4 - 1 = 3$. Besides that, from Equation 1, we get a lower bound on $tvs(P_2 \Box K_n)$ is $\lceil (2n + n)/(n + 1) \rceil = 3$ for $n \geq 3$. Therefore, we have the theorem below.

Theorem 4. *Let P_2 be a path with order 2 and K_n be a complete graph with order n. Then, for $n \geq 3$,*

$$tvs(P_2 \Box K_n) = 3.$$

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